Lesson 6. Markov Chains – Long-Run Probabilities

1 Overview

- Previous lesson: probabilities that depend on the number of time steps, for example

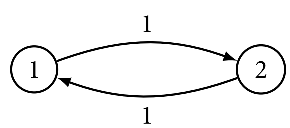
$$p_{ij}^{(n)} = \Pr\{S_n = j \,|\, S_0 = i\} \qquad\qquad q_j^{(n)} = \Pr\{S_n = j\}$$

- This lesson: what happens in the **long run**, i.e. as $n \to \infty$? In particular, what is the **limiting probability**

$$p_{ij}^{(\infty)} = \lim_{n\to\infty} p_{ij}^{(n)} \ \ ?$$

2 Periodic and aperiodic states

- Consider the following two-state Markov chain:



- The n-step transition probability between state 1 and itself is:

- Now consider a Markov chain with state space $\mathcal{M} = \{1, \dots, m\}$
- A state i is **periodic** with period δ (δ is a positive integer) if

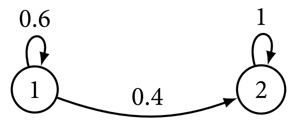
$$p_{ii}^{(n)} \begin{cases} > 0 & \text{if } n = \delta, 2\delta, 3\delta, \dots \\ = 0 & \text{otherwise} \end{cases}$$

and therefore $p_{ii}^{(\infty)} = \lim_{n\to\infty} p_{ii}^{(n)}$ does not exist

- A state i is **aperiodic** if it is not periodic

3 Transient and recurrent states

- Consider the following two-state Markov chain:

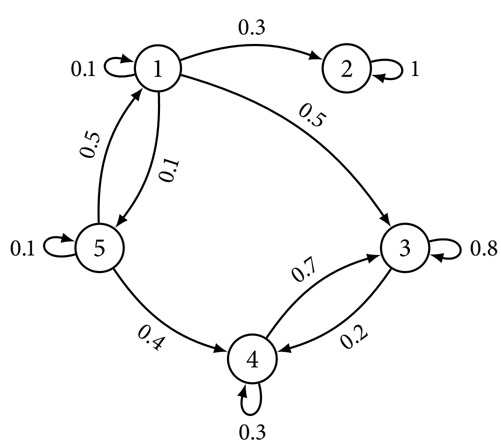


- The limiting probability between state 1 and itself is:

- In other words, the process eventually leaves state 1 and never returns
- Now consider a Markov chain with state space $\mathcal{M} = \{1, \dots, m\}$
- A state i is **transient** if $p_{ii}^{(\infty)} = 0$
  - The process will eventually leave state i and never return
- A state i is **recurrent** if $p_{ii}^{(\infty)} > 0$
  - The process is guaranteed to return to state i over and over again, given that it reaches state i at some time

Example 1. An autonomous UAV has been programmed to move between five regions to perform surveillance, according to a Markov chain in which the states correspond to the regions, and transition probabilities are given below.

Can you guess which states are transient and which states are recurrent?



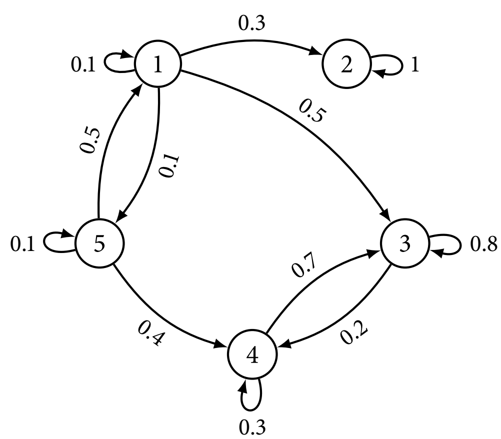
$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0 & 0.1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0.7 & 0.3 & 0 \\ 0.5 & 0 & 0 & 0.4 & 0.1 \end{bmatrix}$$

4 Decomposition of Markov chains

- Can we determine whether a state is transient or recurrent in a systematic way?
- A subset of states $\mathcal{R}$ is a **recurrent class** if
  - $\mathcal{R}$ itself forms a self-contained Markov chain
  - no proper subset of $\mathcal{R}$ also forms a Markov chain
- Once a Markov chain enters a recurrent class, it stays there forever
- A Markov chain is **irreducible** if it has a single recurrent class

Example 2. Consider Example 1 again. The transition probabilities are given below.

Does the subset of states $\mathcal{R}_1 = \{3, 4\}$ form a recurrent class? How about $\mathcal{T} = \{1, 5\}$ or $\mathcal{R}_2 = \{2\}$?
Is the Markov chain irreducible?



$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0 & 0.1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0.7 & 0.3 & 0 \\ 0.5 & 0 & 0 & 0.4 & 0.1 \end{bmatrix}$$

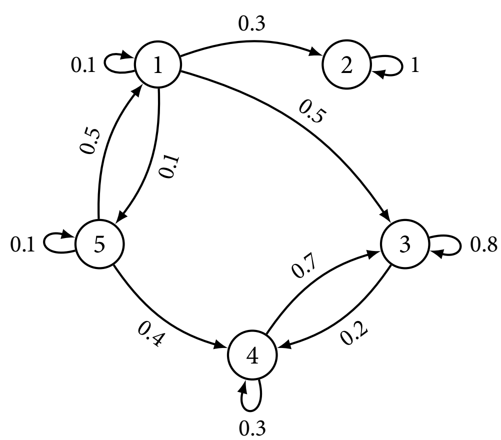
- To find the transient and recurrent states of a Markov chain:
  1. Find all recurrent classes
  2. All states in a recurrent class are recurrent
  3. All states not in a recurrent class are transient

5 Accessible and communicating states

- Briefly, an alternative perspective on the concepts above
- A state j is **accessible** from state i (denoted by $i \rightsquigarrow j$) if
  - there is a sequence of transitions that begins in i and ends in j
  - each transition in the sequence has positive probability
- If $i \rightsquigarrow j$ and $j \rightsquigarrow i$, then states i and j are said to **communicate** (denoted by $i \leftrightsquigarrow j$)
- Some equivalent definitions:
  - A state i is transient iff $i \rightsquigarrow j$ but $j \not\rightsquigarrow i$ for some state j
  - A state i is recurrent iff $j \rightsquigarrow i$ for every state j such that $i \rightsquigarrow j$
  - $\mathcal{R}$ is a recurrent class iff $\mathcal{R}$ forms a self-contained MC and all states communicate with each other

Example 3. Consider Example 1 again. The transition probabilities are given below.

Do all the states in $\mathcal{T} = \{1, 5\}$ communicate with each other? How about $\mathcal{R}_1 = \{3, 4\}$? What is different about the subsets $\mathcal{T}$ and $\mathcal{R}_1$?



$$\mathbf{P} = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0 & 0.1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.8 & 0.2 & 0 \\ 0 & 0 & 0.7 & 0.3 & 0 \\ 0.5 & 0 & 0 & 0.4 & 0.1 \end{bmatrix}$$

6 Limiting probabilities

⚠ Before proceeding, let's take a look at the accompanying Jupyter notebook for this lesson

Example 4. Consider Example 1 again. Recall that the transient states are $\mathcal{T} = \{1, 5\}$, and that there are two recurrent classes, $\mathcal{R}_1 = \{3, 4\}$ and $\mathcal{R}_2 = \{2\}$. In the Jupyter notebook, to approximate long-run behavior, we computed $\mathbf{P}^{1000}$:

$$\mathbf{P}^{1000} = \begin{bmatrix} 0 & 0.3553 & 0.5015 & 0.1433 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0.7778 & 0.2222 & 0 \\ 0 & 0 & 0.7778 & 0.2222 & 0 \\ 0 & 0.1974 & 0.6243 & 0.1784 & 0 \end{bmatrix}$$

What do you notice?

- Suppose we have a Markov chain with state space $\mathcal{M} = \{1, \ldots, m\}$ decomposed into
  - a set of transient states $\mathcal{T}$
  - recurrent classes $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_r$
- We also assume that all states are aperiodic
- Based on how the states are classified, we can compute the limiting probabilities $p_{ij}^{(\infty)}$

6.1 Straightforward cases

- If state j is transient, then $p_{ij}^{(\infty)} =$
  - Since state j is transient, even if we reach j, we will eventually leave j and never return
- If states i and j are in different recurrent classes, then $p_{ij}^{(\infty)} =$
  - State i is one self-contained Markov chain, state j is in another

6.2 Steady-state probabilities

- Suppose states i and j are in the same recurrent class $\mathcal{R}$ with $m_{\mathcal{R}}$ states
- In this case, $p_{ij}^{(\infty)} = \pi_j$ for some $\pi_j > 0$
  - $p_{ij}^{(\infty)}$ in this case does not depend on i!
- The π_j are called **steady-state probabilities**
  - Given that the process reaches the recurrent class containing state j:

$$\begin{aligned} \pi_j &= \text{probability of finding the process in state } j \text{ after a long time} \\ &= \text{the long-run fraction of time that the process spends in state } j \end{aligned}$$

- We can compute π_j by solving the following system of linear equations:

- This system of linear equations has equations and variables
- Using matrix theory, we can show that any one of the equations in $(*)$ is always redundant

Example 5. Consider Example 1 again. Suppose the UAV reaches region 3 at some point. What is the long-run fraction of time that the UAV spends in region 3? Region 4?

6.3 Absorption probabilities

- Suppose state i is transient and state j is the only state in recurrent class $\mathcal{R}$
  - In other words, $\mathcal{R} = \{j\}$ and $p_{jj} =$
  - Such a state is called an **absorbing state**
- In this case, $p_{ij}^{(\infty)} = \alpha_{ij}$ for some $\alpha_{ij} \geq 0$
- The α_{ij} are called **absorption probabilities**
  - What is the probability that the process is ultimately "absorbed" into state j?
- We can compute the α_{ij} using:

Example 6. Consider Example 1 again. Suppose the UAV starts in region 5. What is the probability that the UAV ends up in region 2?

- We can find the probability that the process is ultimately absorbed into a recurrent class $\mathcal{R}$ (possibly with more than 1 state) by lumping the states in $\mathcal{R}$ into a "super state" and then applying the concepts above
  - See Problem 6 in the Exercises

6.4 Expected time to absorption

- The **expected time to absorption** μ_i from state i is
- We can compute the expected time to absorption using:

Example 7. Consider Example 1 again. On average, how long does it take for the UAV to leave regions 1 and 5 forever?

7 Why are the steady-state probabilities computed this way?

- Let $\mathcal{R} = \{1, \dots, m\}$ be a recurrent class
- Why is $\pi_{\mathcal{R}}^\top = \pi_{\mathcal{R}}^\top \mathbf{P}$ used to compute steady-state probabilities?
- By the law of total probability:

$$\Pr\{S_{n+1} = j\} = \sum_{i=1}^{m} \Pr\{S_{n+1} = j \mid S_n = i\} \Pr\{S_n = i\}$$

- In the long run (n very large), $\pi_i = \Pr\{S_{n+1} = i\} = \Pr\{S_n = i\}$, so we get:

$$\pi_j = \sum_{i=1}^{m} p_{ij} \pi_i$$

- This is the equation corresponding to the jth columns of the matrix equation $\pi_{\mathcal{R}}^\top = \pi_{\mathcal{R}}^\top \mathbf{P}$:

$$\begin{bmatrix} \pi_1 & \dots & \boxed{\pi_j} & \dots & \pi_m \end{bmatrix} = \begin{bmatrix} \boxed{\pi_1 \quad \dots \quad \pi_j \quad \dots \quad \pi_m} \end{bmatrix} \begin{bmatrix} p_{11} & \dots & p_{1j} & \dots & p_{1m} \\ \vdots & & \vdots & & \vdots \\ p_{m1} & \dots & p_{m,j} & \dots & p_{mm} \end{bmatrix}$$

8 Exercises

Problem 1 (SMAS Exercise 6.5)**.** Consider a Markov chain with state space $\mathcal{M} = \{1, 2, 3, 4\}$ and transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.5 & 0.4 & 0.1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Classify the states as recurrent or transient.

Problem 2 (SMAS Exercise 6.6)**.** Classify as recurrent or transient the states of the Markov chains with state space $\{1, 2, 3, 4, 5\}$ and the transition probability matrices below by first finding all of the recurrent classes.

$$\text{a. } \mathbf{P} = \begin{bmatrix} 0.1 & 0.3 & 0.4 & 0 & 0.2 \\ 0.5 & 0.1 & 0.1 & 0 & 0.3 \\ 0.8 & 0 & 0 & 0.2 & 0 \\ 0 & 0.1 & 0 & 0.9 & 0 \\ 0.3 & 0.1 & 0.1 & 0 & 0.5 \end{bmatrix} \qquad \text{b. } \mathbf{P} = \begin{bmatrix} 0.1 & 0.5 & 0.1 & 0.1 & 0.2 \\ 0 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0 & 0.7 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \end{bmatrix} \qquad \text{c. } \mathbf{P} = \begin{bmatrix} 0.1 & 0.3 & 0.1 & 0.3 & 0.2 \\ 0 & 0.8 & 0 & 0 & 0.2 \\ 0 & 0 & 0.3 & 0.4 & 0.3 \\ 0 & 1 & 0 & 0 & 0 \\ 0.1 & 0.5 & 0 & 0.4 & 0 \end{bmatrix}$$

Problem 3 (SMAS Exercise 6.8, modified)**.** Consider the Markov chain with state space $\mathcal{M} = \{1, 2, 3, 4, 5, 6\}$ and transition probability matrix

$$\mathbf{P} = \begin{bmatrix} 0 & 0.5 & 0 & 0.4 & 0.1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0.4 & 0.3 & 0 & 0.1 \\ 0 & 0.9 & 0 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

If the process starts in state 3, what is the probability it will be absorbed in state 2? In state 5? In state 6? What is the expected time to absorption from state 3?

Problem 4 (SMAS Exercise 6.11, modified)**.** The Statistical Snacks Company plan to introduce a new cheese snack product, Poisson Puffs, into a local market that already has three strong competitors. The analytics team have formulated a Markov chain model of customer brand switching in which the state space $\mathcal{M} = \{1, 2, 3, 4\}$ corresponds to which of the three established brands or Poisson Puffs, respectively, that a customer currently purchases. Each time step corresponds to one bag of cheese snacks purchased. Based on market research and historical data, the transition probability matrix that the analysts team anticipate is

$$\mathbf{P} = \begin{bmatrix} 0.70 & 0.14 & 0.14 & 0.02 \\ 0.14 & 0.70 & 0.14 & 0.02 \\ 0.14 & 0.14 & 0.70 & 0.02 \\ 0.05 & 0.05 & 0.05 & 0.85 \end{bmatrix}$$

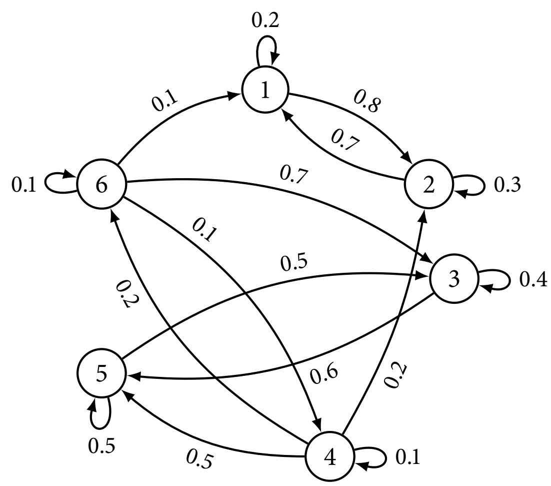
What is the long-term market share for Poisson Puffs?

Problem 5 (SMAS Exercise 6.17)**.** In Problem 1 from Lesson 4, you were asked to develop a Markov chain model for the movement of an automated guided vehicle (AGV) between four locations: a release station A, machining station B, machining station C, and an output buffer D. In this model, the state space was $\mathcal{M} = \{1, 2, 3, 4\}$ (1 = A, 2 = B, 3 = C, 4 = D), each time step represented one AGV trip, and the transition probability matrix was

$$\mathbf{P} = \begin{bmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix}$$

What is the long-run fraction of time the AGV spends traveling to the output buffer?

Problem 6. An autonomous UAV has been programmed to move between six regions to perform surveillance. The movements of the UAV follow a Markov chain with 6 states (1 for each region), and the following transition probability diagram:



a. There are two recurrent classes $\{1, 2\}$ and $\{3, 5\}$. Briefly explain why these sets are recurrent classes.
b. Which states are transient? Which states are recurrent? Briefly explain.
c. Suppose the UAV starts in region 1. What is the long-run fraction of time that the UAV spends in region 1?
d. What is the probability that the UAV is absorbed into states 3 or 5, given that it starts in region 4?